```
C     RETURNS APPROXIMATE SOLUTION VAPX AT X = Z
C=============================================================
C     GLOBAL DECLARATIONS -- SAME IN ALL PROGRAM SEGMENTS
C=============================================================
      PARAMETER (MAXNOD = 30)
      COMMON /PARAMS/ NINPT, NOUTP, IERR
      COMMON /PROBLM/ NODES, RESIST, CONDUC,
     *                X(MAXNOD), VOLTAG(MAXNOD)
      COMMON /MATRIX/ STCON(MAXNOD,MAXNOD), RTHDSD(MAXNOD),
     *      C(MAXNOD,2*MAXNOD-2), STDIS(2*MAXNOD-2,2*MAXNOD-2)
C=============================================================
C
C     DETERMINE IN WHICH INTERVAL Z LIES
      INT = 0
      DO 10 I = 1,NODES-1
      IF (X(I) .LE. Z .AND. X(I+1) .GE. Z) INT = I
   10 CONTINUE
      IF (INT .EQ. 0) GO TO 98
C
C     INTERPOLATE WITHIN INTERVAL TO FIND VALUE
      VAPX = (VOLTAG(INT+1) * (Z - X(INT)) +
     *     VOLTAG(INT) * (X(INT+1) - Z)) / (X(INT+1) - X(INT))
      GO TO 99
C
   98 IERR = 101
   99 RETURN
      END
C
C*************************************************************
C
C     ERROR CODES
C
C*************************************************************
C
C     101   PROBLEM    NODE NUMBER OUT OF ORDER OR OVER MAXIMUM
C     102   PROBLEM    PROBLEM HAS ONLY ONE NODE SO CAN'T SOLVE
C     4xx   DISJOINT   ELEMENT xx HAS ZERO LENGTH -- CHECK DATA
C     701   OUTPUT     FILE WRITING ERROR -- IS FILE OPEN?



   1.         1.         1.
3   2.
```

P.P. Silvester, R.L. Ferrari, Finite Elements For Electrical Engineers, 2nd Edition, Cambridge University Press, 1990.

2

First-order triangular elements for the plane problem

1. Introduction

First-order triangular finite elements made their initial appearance in electrical engineering applications in 1968. They were then used for the solution of comparatively simple waveguide problems, but have since then been employed in many areas where two-dimensional scalar potentials or wave functions need to be determined. Because of their relatively low accuracy, first-order elements have been supplanted in many applications by elements of higher orders. However, they continue to find use in problems where material nonlinearities or complicated geometric shapes are encountered; for example, in analysing the magnetic fields of electric machines, or the charge and current distributions in semiconductor devices.

The first-order methods using triangular elements may be regarded as two-dimensional generalisations of piecewise-linear approximation techniques, tools widely used in virtually all areas of electrical engineering. Furthermore, the mathematics required in defining the elements verges on the trivial, and computer programming at a very simple level can produce many useful results. There are few methods in electromagnetic field theory for which such sweeping claims can be made, and indeed it is surprising that finite elements have not penetrated into electrical engineering applications even more rapidly.

In this chapter, simple, triangular, finite element methods will be developed for solving two-dimensional scalar potential and wave problems. The construction of these simple elements is useful in its own right; but perhaps more importantly, it will also illustrate, by way of example, many of the principles involved in all finite element methods.

2. Laplace's equation

Numerous problems in electrical engineering require a solution of Laplace's equation in two dimensions. For example, determination of the TEM wave properties of a coaxial transmission line composed of rectangular conductors, as in Fig. 2.1(a), requires finding the electric potential distribution in the interconductor space. Only one-quarter of the actual problem region needs to be analysed because of symmetry. Thus, there arise two kinds of boundary conditions in this case: prescribed potential values along the conductive metal surfaces (Dirichlet conditions), and vanishing normal derivative values along the symmetry planes. Subject to these conditions, the potential itself is governed by Laplace's equation,

$$\nabla^2 u = 0, \tag{2.01}$$

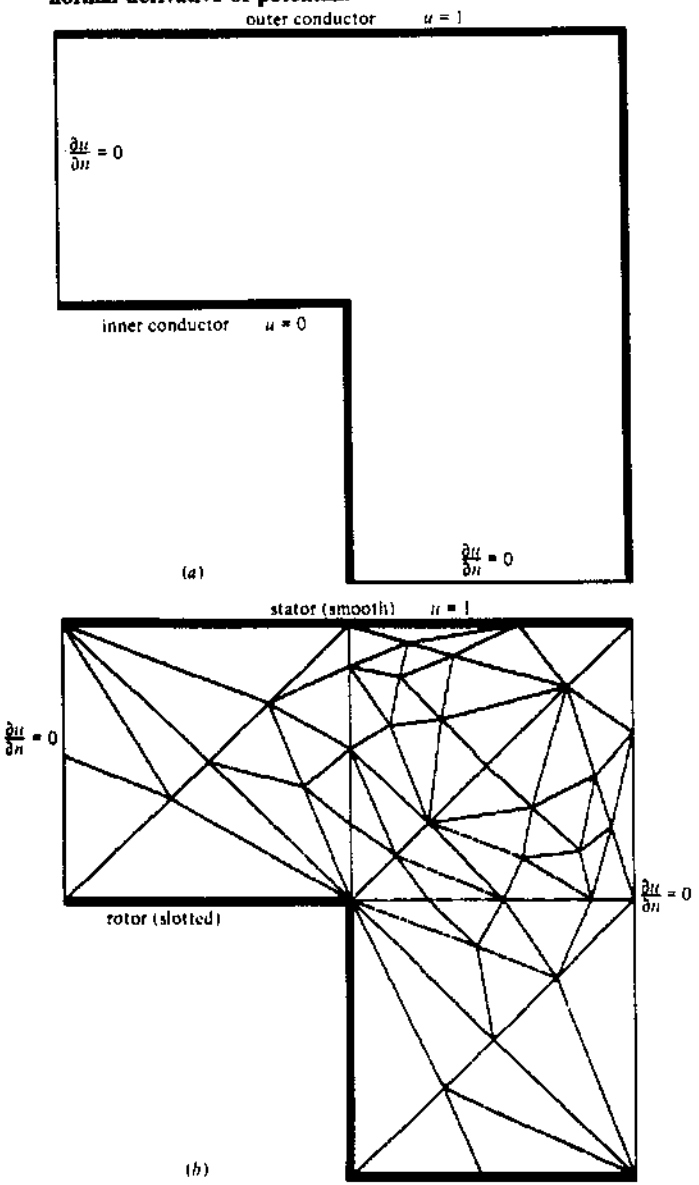
Similarly, classical analysis of electric machines requires determination of the magnetic scalar potential distribution in the air-gap region, as in Fig. 2.1(b). Again, Laplace's equation holds in the interior of this region. The boundary conditions are much like those of the electric potential problem: the scalar potential has fixed values along iron surfaces, and must have a vanishing normal derivative at symmetry planes.

The well-known principle of minimum potential energy requires that the potential distribution in the transmission line or the slot must be such as to minimise the stored field energy per unit length. To within a constant multiplier this energy is given by

$$W(u) = \tfrac{1}{2} \int |\nabla u|^2 \, dS, \tag{2.02}$$

the integration being carried out over the two-dimensional problem region. This minimum-energy principle is mathematically equivalent to Laplace's equation in the sense that a potential distribution which satisfies the latter equation will also minimise the energy, and vice versa. Hence, two alternative practical approaches exist for solving the field problem. On the one hand, an approximate solution to Laplace's equation may be sought directly – as it is, for example, in the separation of variables technique, or in finite difference methods. Alternatively, an approximate expression may be created for the stored energy $W(u)$ associated with the potential $u(x, y)$ by assuming the potential u to be given by a combination of suitably chosen, simple functions with as yet undetermined coefficients. Minimisation of the energy then determines the coefficients, and thereby implicitly determines an approximation to the potential distribution. Virtually all finite element methods follow the second route, or adaptations of it.

Fig. 2.1. (a) One-quarter of the rectangular coaxial line, showing boundary conditions of the problem. (b) Half of one tooth pitch of an electric machine, and its finite element model. The heavy lines indicate Dirichlet boundaries (potential specified), whereas the remaining boundaries are planes of symmetry, requiring vanishing normal derivative of potential.



Suppose $u(x, y)$ is the true solution of the problem, while $h(x, y)$ is some sufficiently differentiable function with exactly zero value at every boundary point where the value of u is prescribed by the boundary conditions. The combination $(u+\theta h)$, where θ is a scalar parameter, then has the same prescribed boundary values as u. The energy $W(u+\theta h)$ associated with this incorrect potential distribution is

$$W(u+\theta h) = W(u) + \theta \int \nabla u \cdot \nabla h \, dS + \tfrac{1}{2}\theta^2 \int |\nabla h|^2 \, dS. \qquad (2.03)$$

The middle term on the right may be rewritten, using Green's theorem (Appendix 2), so that

$$W(u+\theta h) = W(u) + \theta^2 W(h) - \theta \int h \nabla^2 u \, dS + \theta \oint h \frac{\partial u}{\partial n} \, ds. \qquad (2.04)$$

Clearly, the third term on the right vanishes because the exact solution u satisfies Laplace's equation. The last term on the right must also vanish since, at each and every boundary point in Fig. 2.1, either h or the normal derivative of u vanishes. Hence,

$$W(u+\theta h) = W(u) + \theta^2 W(h). \qquad (2.05)$$

The rightmost term in this equation is clearly always positive. Consequently, $W(u)$ is indeed the minimum value of energy, reached when $\theta = 0$ for any admissible function h. Admissibility here implies two requirements: h must vanish at boundary points where u is prescribed; and h must be at least once differentiable.

It is also evident from Eq. (2.05) that the incorrect energy estimate $W(u+\theta h)$ differs from the correct energy $W(u)$ by an error which depends on the square of θ. If the incorrect potential distribution does not differ very greatly from the correct one – that is to say, if θ is small – the error in energy is thus much smaller than the error in potential. This point is of very considerable practical importance, for the quantities actually required by the engineering analyst are often closely related to energy. Impedances, power losses, or the stored energies themselves are often very accurately approximated even if the potential solution contains substantial errors.

3. First-order elements

To construct an approximate solution by a simple finite element method, the problem region is subdivided into triangular elements, as indicated in Fig. 2.1. The essence of the method lies in first approximating the potential u within each element in a standardised fashion, and

thereafter interrelating the potential distributions in the various elements so as to constrain the potential to be continuous across interelement boundaries.

To begin, the potential approximation and the energy associated with it will be built up in this section. Within a typical triangular element, illustrated in Fig. 2.2, it will be assumed that the potential is adequately approximated by the expression

$$U = a + bx + cy. \qquad (3.01)$$

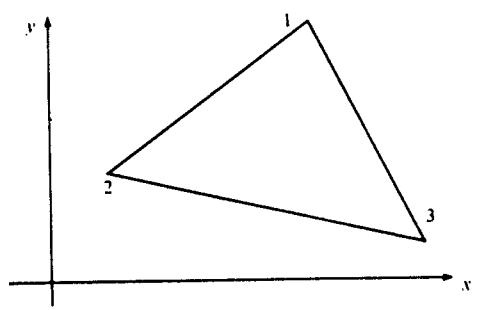
The true solution is thus replaced by a piecewise-planar function; the smoothly-curved, actual potential distribution over the x–y-plane is replaced by a jewel-faceted approximation. It should be noted, however, that the potential along any triangle edge is the linear interpolate between its two vertex values, so that if two triangles share the same vertices, the potential will be continuous across the interelement boundary. There are no gaps in the surface $U(x, y)$ which approximates the true solution over the x–y-plane; the approximate solution is piecewise planar, but continuous everywhere.

The coefficients a, b, c in Eq. (3.01) may be found from the three independent simultaneous equations which are obtained by requiring the potential to assume vertex values U_1, U_2, U_3 at the three vertices. Substituting the three vertex potentials and locations into Eq. (3.01) in turn, there is obtained

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \qquad (3.02)$$

The determinant of the coefficient matrix in Eq. (3.02) may be recognised on expansion as equal to twice the triangle area. Except in the degenerate

Fig. 2.2. Typical triangular finite element in x–y-plane.



case of zero area, the coefficients a, b, c are therefore readily determined by solving the simultaneous equations, Eqs. (3.02). Substitution of the result into Eq. (3.01) then yields

$$U = [1 \quad x \quad y] \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}. \tag{3.03}$$

Combining x, y and the elements of the inverted coefficient matrix into new functions of position, Eq. (3.03) may be written

$$U = \sum_{i=1}^{3} U_i \alpha_i(x, y), \tag{3.04}$$

where

$$\alpha_1 = \frac{1}{2A}\{(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\} \tag{3.05}$$

is a linear function of position only, and A represents the surface area of the triangle. The remaining two functions are obtainable by cyclic interchange of subscripts. It is readily verified from (3.05) that the newly defined functions are *interpolatory* on the three vertices of the triangle, i.e., that each function vanishes at all vertices but one, and that it has unity value at that one:

$$\begin{aligned} \alpha_i(x_j, y_j) &= 0 \quad i \neq j \\ &= 1 \quad i = j. \end{aligned} \tag{3.06}$$

The energy associated with a single triangular element may now be determined using Eq. (2.02), the region of integration being the element itself. The potential gradient within the element may be found from Eq. (3.04) as

$$\nabla U = \sum_{i=1}^{3} U_i \nabla \alpha_i, \tag{3.07}$$

so that the element energy becomes

$$W^{(e)} = \tfrac{1}{2} \int |\nabla U|^2 \, dS. \tag{3.08}$$

or, from (3.07),

$$W^{(e)} = \tfrac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} U_i \int \nabla \alpha_i \cdot \nabla \alpha_j \, dS \, U_j. \tag{3.09}$$

For brevity, define matrix elements

$$S_{ij}^{(e)} = \int \nabla \alpha_i \cdot \nabla \alpha_j \, dS, \tag{3.10}$$

where the superscript identifies the element. Equation (3.09) may thus be written as the matrix quadratic form

$$W^{(e)} = \tfrac{1}{2} \mathbf{U}^{\mathrm{T}} \mathbf{S}^{(e)} \mathbf{U}. \tag{3.11}$$

Here, **U** is the column vector of vertex values of potential; the superscript T denotes transposition.

For any given triangle, the matrix **S** is readily evaluated. On substitution of the general equation, Eq. (3.05), into Eq. (3.10), a little algebra yields

$$S_{12}^{(e)} = \frac{1}{4A}\{(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)\}, \tag{3.12}$$

and similarly for other entries of the matrix **S**.

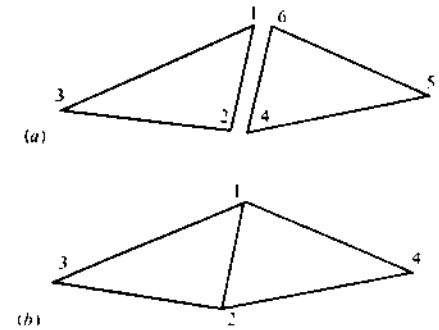
4. Element assembly

For any one triangular element, the element energy may be approximately computed as shown above. The total energy associated with an assemblage of many elements is, in general, the sum of all the individual element energies,

$$W = \sum_{\text{all e}} W^{(e)}. \tag{4.01}$$

Any composite model made up of triangular patches may be built up one triangle at a time. It therefore suffices to consider how continuity is enforced when one triangular element is added to an already existing assemblage. For simplicity, suppose the existing assemblage consists of the one triangle 1–2–3 of Fig. 2.3(a), and the triangle 4–5–6 is to be joined to it. Since three potential values are associated with each triangle,

Fig. 2.3. (a) Adjacent triangular elements, considered to be electrically disjoint. (b) Adjacent triangular elements, with potentials required to be continuous, and nodes renumbered accordingly.



all possible states of the pair of elements are describable by a column vector containing all six vertex potentials,

$$\mathbf{U}_{\text{dis}}^{\text{T}} = [U_1\, U_2\, U_3\, U_4\, U_5\, U_6]_{\text{dis}}, \tag{4.02}$$

where the subscript 'dis' indicates that disjoint elements (elements as yet not joined together in any way) are being considered. The total energy of the pair of elements is then

$$W = \tfrac{1}{2}\mathbf{U}_{\text{dis}}^{\text{T}}\mathbf{S}_{\text{dis}}\mathbf{U}_{\text{dis}}, \tag{4.03}$$

where

$$\mathbf{S}_{\text{dis}} = \begin{bmatrix} S_{11}^{(1)} & S_{12}^{(1)} & S_{13}^{(1)} & & & \\ S_{21}^{(1)} & S_{22}^{(1)} & S_{23}^{(1)} & & & \\ S_{31}^{(1)} & S_{32}^{(1)} & S_{33}^{(1)} & & & \\ & & & S_{44}^{(2)} & S_{45}^{(2)} & S_{46}^{(2)} \\ & & & S_{54}^{(2)} & S_{55}^{(2)} & S_{56}^{(2)} \\ & & & S_{64}^{(2)} & S_{65}^{(2)} & S_{66}^{(2)} \end{bmatrix} \tag{4.04}$$

is the matrix S (the *Dirichlet matrix*) of the disjoint pair of elements. More briefly, in partitioned matrix form,

$$\mathbf{S}_{\text{dis}} = \begin{bmatrix} \mathbf{S}^{(1)} & 0 \\ 0 & \mathbf{S}^{(2)} \end{bmatrix}. \tag{4.05}$$

In the connected assembly of elements, potential values are physically required to be continuous across interelement boundaries. Because the potential in each triangle is approximated by a linear function of x and y, along any one triangle side, its value varies linearly with distance. Hence the continuity requirement on potentials is satisfied, provided the potentials at corresponding vertices are identical. That is to say, the potential in Fig. 2.3(b) will be continuous, if the potentials at vertices 1 and 6 are forced to be exactly equal, and so are the potentials at vertices 2 and 4. Such equality of potentials is implicit in the node numbering for the quadrilateral region, Fig. 2.3(b). Of course, there need not be any particular relationship between the node numbers for the triangles, on the one hand, and the quadrilateral, on the other; the numberings shown in Fig. 2.3 are quite arbitrary. The equality constraint at vertices may be expressed in matrix form, as a rectangular matrix C relating potentials of the disjoint elements to the potentials of the conjoint set of elements (also termed the connected system):

$$\mathbf{U}_{\text{dis}} = \mathbf{C}\mathbf{U}_{\text{con}}, \tag{4.06}$$

where the subscripts denote disjoint and conjoint sets of elements

respectively. With the point numberings shown in Fig. 2.3, this equation takes the form

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix}_{\text{dis}} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \\ 1 & & & \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}_{\text{con}} \tag{4.07}$$

In Eq. (4.07) all those matrix elements have been omitted which must always be zero as a consequence of there being no connection between the corresponding vertices. Substituting Eq. (4.06) into (4.03), the energy for the connected problem becomes

$$W = \tfrac{1}{2}\mathbf{U}_{\text{con}}^{\text{T}}\mathbf{S}\mathbf{U}_{\text{con}}, \tag{4.08}$$

where

$$\mathbf{S} = \mathbf{C}^{\text{T}}\mathbf{S}_{\text{dis}}\mathbf{C} \tag{4.09}$$

represents the assembled coefficient matrix of the connected problem. For the assembly in Fig. 2.3,

$$\mathbf{S} = \begin{bmatrix} S_{11}^{(1)} + S_{66}^{(2)} & S_{12}^{(1)} + S_{64}^{(2)} & S_{13}^{(1)} & S_{65}^{(2)} \\ S_{21}^{(1)} + S_{46}^{(2)} & S_{22}^{(1)} + S_{44}^{(2)} & S_{23}^{(1)} & S_{45}^{(2)} \\ S_{31}^{(1)} & S_{32}^{(1)} & S_{33}^{(1)} & 0 \\ S_{56}^{(2)} & S_{54}^{(2)} & 0 & S_{55}^{(2)} \end{bmatrix} \tag{4.10}$$

The disjoint and conjoint numberings are frequently also termed *local* and *global* numberings respectively.

5. Solution of the connected problem

In the above two sections, the energy of a continuous approximate potential distribution was formulated as a quadratic form involving the column vector of node potentials. To obtain an approximate solution of Laplace's equation, it remains to minimise the stored energy in the connected finite element model. Since the energy expression of Eq. (4.08) is quadratic in the nodal potentials, it must have a unique minimum with respect to each component of the potential vector U. Hence, to minimise it is sufficient to set

$$\frac{\partial W}{\partial U_k} = 0. \tag{5.01}$$

Here the index k refers to entries in the connected potential vector or, what is equivalent, to node numbers in the connected model. Differentiation in Eq. (5.01) with respect to each and every k thus

corresponds to an unconstrained minimisation, with the potential allowed to vary at every node. The unconstrained minimisation, however, does not correspond to the boundary-value problem as originally stated in Fig. 2.1. Indeed, the unconstrained minimum energy is trivially zero, with exactly zero potential everywhere. In the boundary-value problem to be solved, certain boundary segments have specified potential values; thus, a certain subset of the potentials contained in the vector $\mathbf{U}$ must assume exactly those prescribed values. Suppose the node numbering in the connected model is such that all nodes whose potentials are free to vary are numbered first, all nodes with prescribed potential values last. In Fig. 2.1(a), for example, the nodes in the interconductor space (where the potential is to be determined) would be numbered first, and all nodes lying on conductor surfaces (where the potential is prescribed) subsequently. Equation (5.01) may then be written with the matrices in partitioned form,

$$\frac{\partial W}{\partial U_k} = \frac{\partial}{\partial [\mathbf{U}_f]_k} [\mathbf{U}_f^T \mathbf{U}_p^T] \begin{bmatrix} \mathbf{S}_{ff} & \mathbf{S}_{fp} \\ \mathbf{S}_{pf} & \mathbf{S}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{U}_f \\ \mathbf{U}_p \end{bmatrix} = 0, \tag{5.02}$$

where the subscripts f and p refer to nodes with free and prescribed potentials respectively. Note that, since the prescribed potentials cannot vary, differentiation with respect to them is not possible. Performing the differentiation with respect to the free potentials only, there results the rectangular matrix equation

$$[\mathbf{S}_{ff} \quad \mathbf{S}_{fp}] \begin{bmatrix} \mathbf{U}_f \\ \mathbf{U}_p \end{bmatrix} = 0. \tag{5.03}$$

The rectangular coefficient matrix in this equation contains as many rows as there are unconstrained (free) variables; but its number of columns equals the total number of free as well as prescribed nodal potentials. Rewriting, Eq. (5.03) assumes the form

$$\mathbf{S}_{ff}\mathbf{U}_f = -\mathbf{S}_{fp}\mathbf{U}_p. \tag{5.04}$$

The left-hand coefficient matrix is square and, in general, nonsingular; a formal solution to the problem is therefore given by

$$\mathbf{U} = \begin{bmatrix} -\mathbf{S}_{ff}^{-1}\mathbf{S}_{fp}\mathbf{U}_p \\ \mathbf{U}_p \end{bmatrix}. \tag{5.05}$$

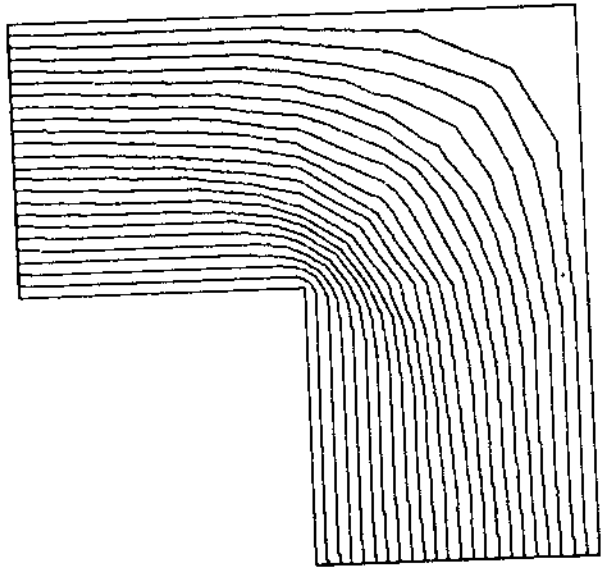
The approximate solution as calculated takes the form of a set of nodal potential values. However, it is important to note that the finite element solution is uniquely and precisely defined everywhere, not only at the triangle vertices, because the energy minimisation assumes the solution surface to have a particular shape. The set of nodal potential values is

merely a compact representation for the piecewise-planar solution surface which yields minimum energy.

Within each triangle the local potential values are prescribed by Eq. (3.01). Thus, no further approximation is necessary to obtain contour plots of equipotential values, to calculate the total stored energy, or to perform any other desired further manipulations. Since in this method the potential in each element is taken to be the linear interpolate of its vertex values, as in Eq. (3.04), an equipotential plot will necessarily consist of piecewise-straight contours. For example, Fig. 1.4 shows equipotential contour plots for the problem of Fig. 1.1(a).

It is worth noting in Fig. 1.4 that the Dirichlet boundary conditions (i.e., boundary conditions requiring the potential to take on prescribed values) are exactly satisfied because the potential values at the boundary nodes are explicitly specified when Eqs. (5.03) and (5.04) are set up. On the other hand, the homogeneous Neumann boundary condition (i.e., the requirement of zero normal derivative) is not satisfied exactly, but only in a certain mean-value sense which causes the contour integral term in Eq. (2.04) to vanish. This boundary condition could of course be rigidly imposed; but the stored field energy would be raised thereby, so that the overall solution accuracy would in fact be worse. This point

Fig. 2.4. Equipotential contours for the coaxial-line problem of Fig. 2.1(a).

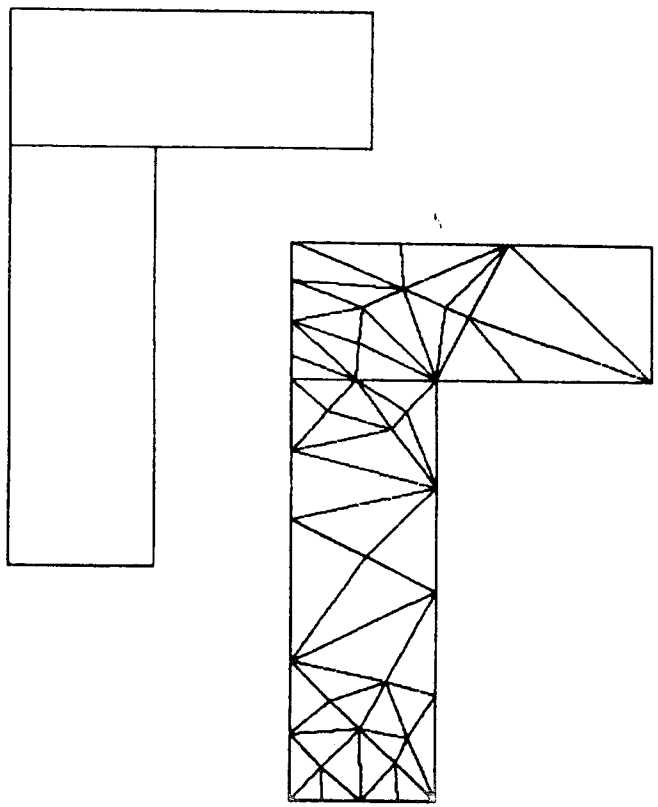


will be considered in some detail later. Roughly speaking, the procedure followed here trades error incurred along the Neumann boundary for an accuracy increase in the solution region.

6. Poisson's equation

Where distributed sources occur within the field region, an approach similar to the above may be used, but with the difference that the source distributions must be explicitly included. As a simple example, Fig. 2.5 shows an electric machine conductor lying in a slot. It can be readily shown that the magnetic field in the slot can be described by the magnetic vector potential A, which satisfies a vector form of the Poisson equation. If the slot and conductor are assumed infinitely long, both the

Fig. 2.5. Electric machine rotor slot and its triangular finite element representation.



current density $\mathbf{J}$ and the magnetic vector potential $\mathbf{A}$ possess only longitudinally directed components; the vector Poisson equation degenerates to its scalar counterpart:

$$\nabla^2 A = -\mu_0 J. \tag{6.01}$$

If the machine iron is assumed infinitely permeable, the normal derivative of A at the slot centreline and at all iron surfaces must be zero. Further, any constant value of A denotes a flux line. Boundary conditions for the problem are thereby clearly defined. The variational problem equivalent to solving Poisson's equation is that of minimising the energy-related functional

$$F(u) = \tfrac{1}{2}\int |\nabla u|^2 \, dS - \mu_0 \int uJ \, dS. \tag{6.02}$$

To show that this functional reaches a minimum at the true solution of Eq. (6.01), suppose A is the correct solution, and v is some differentiable function which vanishes at all boundary points where A is prescribed. Let $F(A+\theta v)$ be evaluated, where θ is a numerical parameter. There is obtained

$$F(A+\theta v) = F(A) + \theta \int \nabla A \cdot \nabla v \, dS - \theta\mu_0 \int vJ \, dS + \tfrac{1}{2}\theta^2 \int |\nabla v|^2 \, dS. \tag{6.03}$$

Using Green's theorem (Appendix 2), the second term on the right may be changed to read

$$\int \nabla A \cdot \nabla v \, dS = \oint v \frac{\partial A}{\partial n} ds - \int v \nabla^2 A \, dS. \tag{6.04}$$

The contour integral on the right vanishes since either v or the normal derivative of A is zero at every point along the boundary. Since A is the correct solution of Eq. (6.01), it is further possible to rewrite Eq. (6.04), altering the right-hand surface-integral term, as

$$-\int v \nabla^2 A \, dS = \mu_0 \int vJ \, dS. \tag{6.05}$$

The functional of Eq. (6.03) thus simplifies to

$$F(A+\theta v) = F(A) + \tfrac{1}{2}\theta^2 \int |\nabla v|^2 \, dS. \tag{6.06}$$

Since the integral on the right is always positive, it is evident that a minimum will be reached when θ has zero value; and conversely, that $F(u)$ reaches its minimum value for $u = A$, the solution of Eq. (6.01).

It will be noted that the field energy is still calculable by the general expression

$$W = \tfrac{1}{2}\int |\nabla A|^2 \, dS \tag{6.07}$$

or, alternatively, by evaluating the equivalent expression

$$W = \frac{\mu_0}{2}\int AJ \, dS. \tag{6.08}$$

At its minimum value $F(A)$, F evidently has a negative value equal in magnitude to the total stored energy. Once again, the error term in Eq. (6.06) depends on the square of the parameter θ. Near the correct solution, θ is small. The accuracy with which the stored energy can be found is therefore very high, even if the potential values are locally not very accurate.

7. Modelling the source term

To construct a finite element model of the Poisson-equation problem, a procedure will be employed similar to that used for Laplace's equation. The problem region will again be triangulated, as shown in Fig. 2.5 and, initially, a typical triangular element will be examined in isolation from the rest. Since the first term in the functional of Eq. (6.02) is identical to the right-hand side of Eq. (2.02), the discretisation process follows exactly the same steps, and leads to exactly the same result for the connected model, Eq. (4.08). The second term in Eq. (6.02) requires treatment which is similar in principle, but slightly different in details.

Over any one triangle, the prescribed current density $J(x, y)$ will be approximated in a manner similar to the potential

$$J(x, y) = \sum_{i=1}^{3} J_i \alpha_i(x, y), \tag{7.01}$$

where the right-hand coefficients are vertex values of current density within the triangle. These values are of course known, since the current density itself is a prescribed function. The source integral may therefore be written

$$\int AJ \, dS = \sum_{i=1}^{3}\sum_{j=1}^{3} A_i \int \alpha_i \alpha_j \, dS \, J_j, \tag{7.02}$$

with the vertex potential values the only unknowns. For each element, let another square matrix of order 3 be defined by

$$T_{ij}^{(e)} = \int \alpha_i \alpha_j \, dS, \tag{7.03}$$

so that

$$\int AJ \, dS = \mathbf{A}^{\mathrm{T}}\mathbf{T}^{(e)}\mathbf{J}, \tag{7.04}$$

where the superscript e identifies the element, and the region of integration is understood to be that element.

For the disjoint set of triangular elements, the functional of Eq. (6.02) now becomes

$$F(A) = \tfrac{1}{2}\mathbf{A}_{\mathrm{dis}}^{\mathrm{T}}\mathbf{S}_{\mathrm{dis}}\mathbf{A}_{\mathrm{dis}} - \mu_0\mathbf{A}_{\mathrm{dis}}^{\mathrm{T}}\mathbf{T}_{\mathrm{dis}}\mathbf{J}_{\mathrm{dis}}. \tag{7.05}$$

The element interconnection once again expresses itself in the requirement of potential continuity and hence in a constraint transformation like Eq. (4.06). Thus

$$F(A) = \tfrac{1}{2}\mathbf{A}^{\mathrm{T}}\mathbf{S}\mathbf{A} - \mu_0\mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{T}_{\mathrm{dis}}\mathbf{J}_{\mathrm{dis}}. \tag{7.06}$$

Minimisation of $F(A)$ with respect to each and every unconstrained vertex potential, putting

$$\frac{\partial F}{\partial A_k} = 0, \tag{7.07}$$

leads to the matrix equation

$$\mathbf{S}'\mathbf{A} = \mu_0\mathbf{C}^{\mathrm{T}}\mathbf{T}_{\mathrm{dis}}\mathbf{J}_{\mathrm{dis}} \tag{7.08}$$

as the finite element model of the boundary-value problem. In general, there is no need for source densities to be continuous across interelement boundaries. Therefore, no further transformations need apply to the right-hand side, except in particular cases.

Since differentiation cannot be carried out in Eq. (7.07) with respect to fixed potentials, the matrix $\mathbf{S}'$ of Eq. (7.08) is rectangular. It possesses as many rows as there are unconstrained nodes in the problem, and as many columns as there are nodes in the model. Just as in Eq. (5.03), the vector of node potentials will now be partitioned so as to include all unconstrained potentials in the upper, all prescribed potentials in its lower part. The matrix $\mathbf{S}'$ is partitioned conformably, leading to

$$[\mathbf{S}_{\mathrm{ff}} \quad \mathbf{S}_{\mathrm{fp}}]\begin{bmatrix}\mathbf{A}_{\mathrm{f}}\\ \mathbf{A}_{\mathrm{p}}\end{bmatrix} = \mu_0\mathbf{C}^{\mathrm{T}}\mathbf{T}_{\mathrm{dis}}\mathbf{J}_{\mathrm{dis}}. \tag{7.09}$$

As before, the subscripts f and p refer to free and prescribed potential values respectively. Since the latter values are known, they will be moved to the right-hand side:

$$\mathbf{S}_{\mathrm{ff}}\mathbf{A}_{\mathrm{f}} = \mu_0\mathbf{C}^{\mathrm{T}}\mathbf{T}_{\mathrm{dis}}\mathbf{J}_{\mathrm{dis}} - \mathbf{S}_{\mathrm{fp}}\mathbf{A}_{\mathrm{p}}. \tag{7.10}$$

Solution of this equation determines the unknown nodal potential values, thus solving the problem.

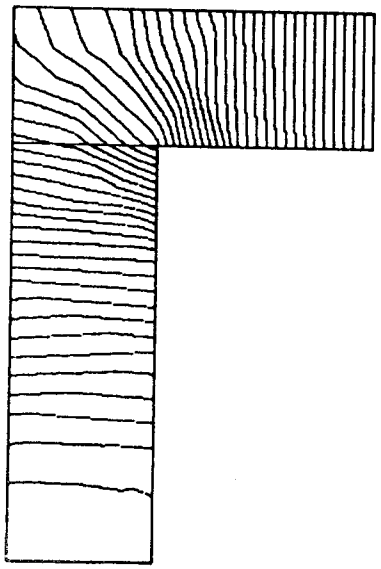
It is interesting to note that the right-hand side of Eq. (7.10) combines the source term of the differential equation (the inhomogeneous part of the equation) with the effect of prescribed boundary values (i.e., the inhomogeneous part of the boundary conditions). Thus there is no fundamental distinction between the representation of a homogeneous differential equation with inhomogeneous boundary conditions, on the one hand, and an inhomogeneous differential equation with homogeneous boundary conditions, on the other.

A solution, using the rather simple triangulation of Fig. 2.5, appears in Fig. 2.6. Just as in the case of Laplace's equation, the Dirichlet boundary conditions (flux line boundary conditions) are rigidly enforced, while the Neumann boundary conditions are not. As can be seen in Fig. 2.6, the latter conditions are therefore locally violated, but satisfied in the mean.

8. Practical handling of boundary conditions

The finite element problems as set up in Eqs. (5.03) and (7.09) necessitate numbering the problem variables in a special fashion. To arrange the equations in the manner shown, all fixed potentials must be numbered last, the potentials free to vary must be numbered first. In

Fig. 2.6. Solution of the electric machine slot problem of Fig. 2.5.



practice, it is not always convenient to renumber variables, nor to partition matrices in this way. Happily, the renumbering and partitioning are only required for purposes of explanation; in practical computing they are never necessary.

Consider again the very simple two-element problem shown in Fig. 2.3(b). It is assumed that potentials 3 and 4 are fixed, 1 and 2 free to vary; in other words, the variable numbering is fully in accordance with the scheme used above. Following (5.04), the matrix equation to be solved then has the form

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = - \begin{bmatrix} S_{13} & S_{14} \\ S_{23} & S_{24} \end{bmatrix} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix}. \tag{8.01}$$

There is not much to be said regarding the high-numbered potentials, which are constrained to have fixed values. Thus the only equation that can be written to describe them is a general form of the identity

$$\mathbf{D} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} D_{33} & \\ & D_{44} \end{bmatrix} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix}, \tag{8.02}$$

where $\mathbf{D}$ is any square, diagonal matrix. Equation (8.02) simply says that the fixed potentials are what they are. It may be combined with Eq. (8.01):

$$\begin{bmatrix} S_{11} & S_{12} & & \\ S_{21} & S_{22} & & \\ & & D_{33} & \\ & & & D_{44} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} -S_{13} & -S_{14} \\ -S_{23} & -S_{24} \\ D_{33} & \\ & D_{44} \end{bmatrix} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix}. \tag{8.03}$$

Next, let an arbitrary numbering be introduced for the potentials, one which does not necessarily take the variable and fixed potentials in any particular sequence. For example, let the vertices 1–2–3–4 be renumbered 2–4–1–3, so that the fixed potentials now reside at nodes 1 and 3. The physical problem obviously does not change in any way as a result of the renumbering; only the rows and columns of the coefficient matrix in Eq. (8.03) are permuted into a new sequence in keeping with the new numbering. Thus (8.03) takes the form

$$\begin{bmatrix} D_{11} & & & \\ & S_{22} & & S_{24} \\ & & D_{33} & \\ & S_{42} & & S_{44} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} D_{11} & \\ -S_{21} & -S_{23} \\ & D_{33} \\ -S_{41} & -S_{43} \end{bmatrix} \begin{bmatrix} U_1 \\ U_3 \end{bmatrix}. \tag{8.04}$$

The matrix elements left blank in the above are all zero as a consequence of the problem structure, independently of physical dimensions or material properties.

While Eq. (8.04) has more rows and columns than (8.01), so that the cost of solution is higher, there is no need for any particular numbering of vertices and potentials; they may be numbered as desired. Thus the increased cost of handling the matrix problem is at least partly compensated by the work saved in not renumbering and rearranging equations.

In practice, the diagonal matrix **D**, above, is often taken to be the unit matrix **D** = **I**. Occasionally, matrices **S** are encountered whose entries are very large or very small compared to unity, so that numerical roundoff-error considerations may dictate a different choice of **D**. For example, if the matrix elements in (8.01) are typically of the order of 1.0E−10, they may be lost, compared with unity, on a computer capable of seven-digit arithmetic. Fortunately, such circumstances do not often arise, so that **D** = **I** is very frequently used. In such a case, (8.04) becomes simply

$$\begin{bmatrix} 1 & & & \\ & S_{22} & & S_{24} \\ & & 1 & \\ & S_{42} & & S_{44} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 1 & \\ -S_{21} & -S_{23} \\ & 1 \\ -S_{41} & -S_{43} \end{bmatrix} \begin{bmatrix} U_1 \\ U_3 \end{bmatrix}. \tag{8.05}$$

Equation (8.05) implies that setting up the finite element equations and imposing the boundary conditions can conveniently be done at the same time, and on an element-by-element basis. As each element matrix is constructed, row and column numbers are scanned to determine whether they correspond to free or fixed potentials. Matrix entries which correspond to free potentials are entered in the natural fashion. Fixed potential values, on the other hand, are treated by substituting rows and columns of the unit matrix on the left, and by augmenting the right-hand side.

9. Programming and data structures

A major strength of the finite element method, even using first-order triangular elements, resides in its great geometric flexibility. Unlike many other numerical methods, the finite element technique is not strongly restricted in the geometrical shapes which may be treated. Using triangular elements, for example, any two-dimensional region may be treated whose boundary can be satisfactorily approximated by a series of straight-line segments. It should also be noted that the triangular-element mesh by which the interior of the problem region is modelled, is regular neither geometrically nor topologically, the triangles being of varying sizes and shapes, while their interconnection does not necessarily follow a regular pattern.

Approximate solution of a given physical problem by means of finite elements may be regarded as comprising five distinct stages:

(i) creation of finite element mesh, i.e., subdivision of the problem region into elements;
(ii) definition of the sources and imposed boundary values of the problem;
(iii) construction of the matrix representation of each element;
(iv) assembly of all elements, by matrix transformations such as Eq. (4.09), and imposition of boundary conditions;
(v) solution of the resulting simultaneous algebraic equations;
(vi) display and evaluation of the results.

In essence, the geometrically and mathematically complicated boundary-value problem is described as a disjoint set of elements in the first stage; all the subsequent stages serve to reassemble the pieces in a systematic fashion so as to produce the desired solution. The middle three stages clearly involve numerical work of a repetitive and highly systematic character, and are thus ideally suited to the digital computer. The matrix representation of each triangular element can be carried out provided only that the vertex locations of that one triangle are known, without any knowledge of the nature of the entire mesh. Conversely, assembly and imposition of boundary conditions only require knowledge of the mesh topology, i.e., of the manner in which the triangles are interconnected.

Assembly of all individual element matrices to form the global matrix representation requires the connection transformation, Eqs. (4.05)–(4.06), to be executed. All the required topological information is contained in the connection matrix **C**. However, it would clearly be most inefficient to store the connection matrix in the explicit form of Eq. (4.07), and the disjoint global matrices in the form given by Eq. (4.04), for both matrices have rather special form and both contain a high proportion of zero entries. The key to efficient assembly is furnished by Eq. (4.10). The latter may be rewritten in the form

$$\mathbf{S} = \begin{bmatrix} S_{11}^{(1)} & S_{12}^{(1)} & S_{13}^{(1)} & \\ S_{21}^{(1)} & S_{22}^{(1)} & S_{23}^{(1)} & \\ S_{31}^{(1)} & S_{32}^{(1)} & S_{33}^{(1)} & \\ & & & \end{bmatrix} + \begin{bmatrix} S_{66}^{(2)} & S_{64}^{(2)} & S_{65}^{(2)} \\ S_{46}^{(2)} & S_{44}^{(2)} & S_{45}^{(2)} \\ & & \\ S_{56}^{(2)} & S_{54}^{(2)} & S_{55}^{(2)} \end{bmatrix}, \tag{9.01}$$

which suggests that, in the general case, the assembled **S**-matrix may be obtained by calculating each individual element matrix, then immediately adding the nine nonzero matrix contributions due to that element to the corresponding entries in the global **S**-matrix. This technique suggests in turn that a very compact and convenient method of storing the

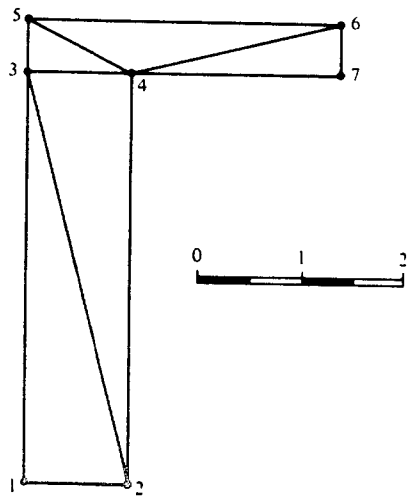
topological information contained in **C** will be by means of an array that identifies the three vertices of each triangle in terms of their global node numbers.

To proceed by way of example, consider the very simple mesh representation of Fig. 2.7 for the slot-conductor problem. The N triangles that constitute the problem region can be identified by a 3-by-N array giving the node numbers, and a 1-by-N array of the source densities in the elements. These arrays may be read in from an input device (card reader, keyboard, etc.). With one element per input line, data for this problem then appear as follows:

```
1   2   3   1.000
2   4   3   1.000
3   4   5   0.000
4   7   6   0.000
4   6   5   0.000
```

The assembly then proceeds by zeroing the matrix **S**, forming the element matrix for triangle 1–2–3, adding the nine numbers thus generated to **S**, then computing the element matrix for triangle 2–4–3, adding to **S**, ... until the entire matrix **S** has been assembled. Computing the matrix representation of each individual element requires knowledge of

Fig. 2.7. A very simple model for the slot problem, to illustrate data handling techniques.



the coordinates of its vertices. For this purpose, 1-by-M arrays of the x- and y-coordinates are needed, where M is the number of nodes:

```
0.000   0.000
1.000   0.000
0.000   4.000
1.000   4.000
0.000   4.500
3.000   4.500
3.000   4.000
```

Finally, the boundary conditions must be entered. For this purpose, it suffices to specify boundary-point numbers and corresponding boundary values:

```
6   0.000
7   0.000
```

The resulting program, implemented in the Fortran language, is given in Section 10 below. It will be noted that the program itself contains no information specific to the particular problem being solved. All the geometric, topological, and other problem-dependent information is contained in data arrays, which are prepared separately and read in when required. Another quite different problem in Poisson's equation can be solved using exactly the same program, by supplying a different file of data.

10. A sample program

The methods set out above are embodied in the Fortran program shown in Section 12. Only the very simplest of methods are employed in this program, in an attempt to keep it easy to read and easy to modify. The overall program structure, on the other hand, is similar to most of the very large, complex programs now in industrial and research use.

The finite element package described here comprises a main program and various subroutines. In addition to these, it also calls upon subroutines contained in the general-purpose library package of Appendix 3. The main program contains no executable statements other than subroutine calls. It communicates with subroutines through COMMON blocks which reappear identically in all the subroutines. In other words, the arrays and other data items are placed in a storage area accessible to all program segments alike. The practical significance of this data arrangement is important: if any alteration is required (for example, if array dimensions are to be changed) it is only necessary to restructure the COMMON block appropriately, and then to replace the COMMON block in every

subroutine with an identical copy of the new version. No other reprogramming, and no extensive program testing, will be required.

The main program shown here is purely a steering mechanism; it defines the data structure and sequences the subroutine calls, but it does not actually perform any computations. Such a program organisation is common, for it allows individual subroutines to be replaced without affecting the operation of the others. For example, partial differential equations other than Poisson's equation could be solved by the program shown here, provided the element matrices used were appropriate to the new differential equation. But since the element matrices are generated by one subroutine, which does not carry out any other operations, the only program alteration necessary consists of removing that one subroutine and replacing it with a different one. The other program functions, such as data input, finite element assembly and equation solving, are all unaffected. For example, axisymmetric problems can be tackled with this program, provided an axisymmetric element subroutine is substituted for the x–y version shown below.

The subroutines called by the main program, in the order of their appearance, are:

MESHIN reads in and stores the problem data. It does a very limited amount of validity checking. More sophisticated programs of this type often differ in the amount of data verification performed. They are often the largest single subroutines in finite element programs because the number of possible mistakes in a data set is very large. Are there any overlapping triangles? Have all the triangle vertices been properly declared as nodes? Are they all geometrically distinct? Is the problem ill-posed (e.g., all potentials fixed)? Is the problem solvable (e.g., are there any fixed boundary values at all)? Are there any conflicts in boundary conditions (e.g., same node assigned two values)?

MATINIT and VECINIT initialise the global element matrices to zero value.

ELMATR computes the matrix representation of one first-order triangular element. It uses techniques discussed in detail in Chapter 4, which differ slightly from those given above.

ELEMBD embeds the matrix contributions of one element into the global coefficient matrix and right-hand side.

EQSOLV solves simultaneous algebraic equations, so as to obtain the potentials not known in advance. It assumes that fixed and free potentials may have been intermixed, as discussed in Section 8 above, so that it is not necessary to precede EQSOLV by equation and variable renumbering and rearrangement. EQSOLV is listed in Appendix 3 and is described in detail in Chapter 10.

OUTPUT prints out the solution once it has been obtained. This routine is very complicated in many finite element programs; it often incorporates graphic plotting, calculation of total stored energy, determination of maximal field values, and many other quantities of interest. Here it has been reduced to the most elementary form conceivable, a neatly formatted dump of all known values.

Although simple in concept, finite element methods lead to relatively complex program structures if reasonable flexibility in problem geometry and range is desired. On the other hand, even at the simplest level, finite element programs can (and should) be written to be highly problem-independent. It is important that the user of finite element methods acquires some understanding of the program structure involved, as well as an appreciation of the underlying mathematics. However, it is usually unwise to plunge into *ad hoc* program development for a specific problem – the development of efficient and error-free programs is a complex task often best left to specialists. The user primarily interested in solving particular problems will often find his effort better invested in modifying already existing programs, rather than in succumbing to the temptation to start over again from scratch.

11. Annotated bibliography

The finite element method using first-order elements has been applied to a large variety of electrical engineering problems in the past, and will no doubt continue to be applied. Although first-order elements do not produce solutions of high accuracy, the method is simple to understand, simple to program, and above all simple to formulate where the fundamental physical equations are more complicated than those illustrated in this chapter.

The first application of triangular elements to the calculation of electric or other potential fields was probably that of Courant (1943). In his paper, piecewise-approximation methods similar to finite elements were first developed. First-order triangular elements in essentially their present form were developed by Duffin (1959), who indicated the

methods for solution and pointed out the availability, not merely of approximate solutions but of bounds for the stored field energy.

There are many monographs and textbooks on finite element theory, as applied to structural engineering. Unfortunately, these are rarely easy for the electrical engineer to read. The text by Norrie & de Vries (1978) is written from the viewpoint of a mechanical engineer, but is sufficiently interdisciplinary to satisfy some electrical engineers as well. The little book by Owen & Hinton (1980) is easy to read, but many will find it not really sufficient for it covers little more than the content of this chapter.

The paucity of textbook material in this field, however, is richly compensated by articles in periodicals and conference records. With the growing popularity of finite element methods for structural analysis in the 1960s, Zienkiewicz & Cheung (1965) attempted solution of practical potential problems and reported on their results. First-order triangular elements were applied to electrical problems shortly thereafter by Silvester (1969), as well as by Ahmed & Daly (1969).

The geometric flexibility of first-order triangles has endeared them to many analysts, so that they are at times used even where higher-order elements would probably yield better accuracy with lower computing costs. Andersen (1973) gives examples of well-working computer programs using first-order elements for daily design work.

In principle, circuits are mathematical abstractions of physically-real fields; nevertheless, electrical engineers at times feel they understand circuit theory more clearly than fields. Carpenter (1975) has given a circuit interpretation of first-order elements.

Particularly if iterative equation-solving methods are employed, first-order finite element techniques sometimes resemble classical finite difference methods. The question is occasionally asked as to which method should then be preferred. This problem was investigated by Demerdash & Nehl (1976), who compared results for sample problems solved by both methods. Finite elements seem preferable even at the first-order level.

An interesting point to note about all finite element methods is that the approximate solution is uniquely defined at all points of interest, not merely on certain discretely chosen points as in finite difference methods. Further use of the solutions, therefore, often requires no further approximation. For example, Daly & Helps (1972) compute capacitances directly from the finite element approximation, without additional assumptions. Similarly, Bird (1973) estimates waveguide losses by reference to the computed solution.

Ahmed, S. & Daly, P. (1969). 'Waveguide solutions by the finite element method', *Radio and Electronic Engineer*, **38**, 217–23.

Andersen, O. W. (1973). 'Transformer leakage flux program based on finite element method', *Institute of Electrical and Electronics Engineers Transactions on Power Apparatus and Systems*, **PAS–92**, 682–9.

Bird, T. S. (1973). 'Evaluation of attenuation from lossless triangular finite element solutions for inhomogeneously filled guiding structures', *Electronics Letters*, **9**, 590–2.

Carpenter, C. J. (1975). 'Finite element network models and their application to eddy-current problems', *Institution of Electrical Engineers Proceedings*, **122**, 455–62.

Courant, R. L. (1943). 'Variational method for the solution of problems of equilibrium and vibration', *Bulletin of the American Mathematical Society*, **49**, 1–23.

Daly, P. & Helps, J. D. (1972). 'Direct method of obtaining capacitance from finite element matrices', *Electronics Letters*, **8**, 132–3.

Demerdash, N. A. & Nehl, T. W. (1976). 'Flexibility and economics of the finite element and difference techniques in nonlinear magnetic fields of power devices', *Institute of Electrical and Electronics Engineers Transactions on Magnetics*, **MAG–12**, 1036–8.

Duffin, R. J. (1959). 'Distributed and lumped networks', *Journal of Mathematics and Mechanics*, **8**, 793–826.

Norrie, D. H. & de Vries, G. (1978). *An Introduction to Finite Element Analysis*. New York: Academic Press.

Owen, D. R. J. & Hinton, E. (1980). *A Simple Guide to Finite Elements*. Swansea: Pineridge Press.

Silvester, P. (1969). 'Finite element solution of homogeneous waveguide problems', *Alta Frequenza*, **38**, 313–17.

Zienkiewicz, O. C. & Cheung, Y. K. (1965). 'Finite elements in the solution of field problems', *The Engineer*, Sept. 24, 1965, pp. 507–10.

12. Programs

Program SIMPLE2D, reproduced below, is closely related to other programs in this book and shares various utility subroutines listed in Appendix 3. Apart from these, it is self-contained. It is followed by a simple data set which should permit testing for correct compilation and execution.

```
C*************************************************************
C************                                     ************
C************  FIRST-ORDER DEMONSTRATION PROGRAM  ************
C************                                     ************
C*************************************************************
C*************************************************************
C     COPYRIGHT (C) 1988 P. P. SILVESTER AND R. L. FERRARI
C*************************************************************
C
C       SOLVES THE  PROBLEM IN FILE "NINPT" AND  WRITES RESULTS
C       TO FILE "NOUTP"; FILE NAMES ARE ASSIGNED INTERACTIVELY.
C
C       THE SUBROUTINES  THAT MAKE UP THIS PROGRAM  COMMUNICATE
C       VIA NAMED COMMON BLOCKS.  THE VARIABLES IN COMMONS ARE:
```